Topology and its Applications 158 (2011) 594–596

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On chaotic extensions of dynamical systems

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A R T I C L E   I N F O

Article history:
Received 25 February 2010
Received in revised form 11 November 2010
Accepted 11 November 2010

MSC:
54H20
37B05
37B20

Keywords:
Extensions
Chaotic
Metric space

A B S T R A C T

In this paper, inspired by some results in linear dynamics, we will show that every dynamical system (X, f), where f is a continuous self-map on a separable metric space X, can be extended to a chaotic (in the sense of Devaney) dynamical system in an isometric way.

1. Introduction and preliminaries

A pair (X, f), where f is a continuous self-map on a metric space (X, d), will be called a dynamical system.

Let (X, f) and (Z, F) be dynamical systems. Following the terminology of [5], we say that (Z, F) is an extension of (X, f) if there exists a homeomorphic embedding $\varphi : X \to Z$ such that $F \circ \varphi = \varphi \circ f$.

If φ is an isometry, we say that (Z, F) is an isometric extension of (X, f).

$\mathbb{N}$ and $\mathbb{N}_0$ will denote the sets of positive integers and nonnegative integers, respectively.

According to Devaney [2] we say that a dynamical system (X, f) (or simply the map f) is chaotic if:

(i) f is (topologically) transitive, i.e., for every pair U and V of non-empty open subsets of (X, d) there is some $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$;
(ii) f is periodically dense, i.e., the set of periodic points of f is dense in X;
(iii) f has sensitive dependence on initial conditions, i.e., there is some $\delta > 0$ such that, for any $x \in X$ and any neighbourhood V of x in (X, d), there exist $y \in V$ and some $\kappa \in \mathbb{N}_0$ for which $d(f^\kappa(x), f^\kappa(y)) > \delta$.

It is worth noting that, in the definition above, condition (iii) is redundant if X is infinite [1,8].

The aim of this paper is to show that every dynamical system (X, f), where X is a separable space, admits an isometric chaotic extension.

Our work is motivated by the following results in linear dynamics:

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2010.12.007

(1) [5] Let T be a bounded linear operator on a Banach (Hilbert) space X. Then (X, T) has an isometric extension $(\tilde{X}, \tilde{T})$, where X is a Banach (Hilbert) space and $\tilde{T}$ is a chaotic bounded linear operator on $\tilde{X}$.
(2) [5] Let X be a metric space and $f : X \to X$ a Lipschitz map. Then (X, f) admits an isometric extension $(\tilde{X}, \tilde{f})$, where $\tilde{X}$ is a Banach space and $\tilde{f}$ is a chaotic bounded linear operator on $\tilde{X}$.
(3) [4] Let f be a continuous self-map on a compact metric space X. Then (X, f) admits an extension $(\tilde{X}, \tilde{f})$, where $\tilde{X}$ is a Hilbert space and $\tilde{f}$ is a chaotic bounded linear operator on $\tilde{X}$.

It is worth pointing out that the extension, in the last result, is not isometric.

We refer the reader to [3] and [6] for notations and terminology not explicitly given.

2. The results

Let us start with a general construction and a subsequent lemma which will play a key role in our main result.

Let (X, d_0) be a metric space, set $X_0 = X$ and let $\varepsilon_n > 0$ for every $n \in \mathbb{N}$. Moreover, for every $n \in \mathbb{N}_0$, let:

(i) $X_{n+1} = X_n \cup \{q_{n+1}\}$, where $q_{n+1} \notin X_n$;
(ii) $b_{n+1} \in X_n$.

Let d_n, for every $n \in \mathbb{N}$, be the metric on X_n given by $d_n|X_{n-1} = d_{n-1}$ and $d_n(q_n, y) = \varepsilon_n + d_{n-1}(b_n, y)$ for every $y \in X_{n-1}$. Observe that, in particular, $d_n(q_n, b_n) = d_n(q_n, X_{n-1}) = \varepsilon_n$.

Now set $Z = \bigcup_{n\in\mathbb{N}_0} X_n$ and let $d = \bigcup_{n\in\mathbb{N}_0} d_n$ be the metric on Z.

Lemma 1. *Let (X, d_0) and (Z, d) be as above, with $\varepsilon_{n+1} \leqslant 2\varepsilon_n$ whenever $\varepsilon_n < 1$.*
Moreover let $f : (Z, d) \to (Z, d)$ be a map such that:

(i) *$f|X$ is continuous;*
(ii) *$f(q_n) = q_{n+1}$, $f(b_n) = b_{n+1}$.*

Then f is continuous.

Proof. Let $x \in Z$. To verify that f is continuous at x, it is enough to show that for every $\varepsilon \in\,]0, 1[$ there exists some $\delta_0 > 0$ such that $d(f(x), f(z)) < 3\varepsilon$ whenever $d(x, z) < \delta_0$.

Now let $\varepsilon \in\,]0, 1[$, and let $\kappa \in \mathbb{N}$ be such that $x, f(x) \in X_\kappa$. Since every point of $X_\kappa \setminus X$ is isolated in X_κ, it follows that $f|X_\kappa$ is continuous, so there is some $\delta_0 \in\,]0, \varepsilon[$ such that $d(f(x), f(z)) < \varepsilon$ whenever $d(x, z) < \delta_0$ and $z \in X_\kappa$.

Let us show, by induction, that for every $n \geqslant \kappa$ and every $\delta \in\,]0, \delta_0]$ the following holds:

$$d\bigl(f(x), f(z)\bigr) < \varepsilon + 2\delta \quad \text{whenever } z \in X_n \text{ and } d(x, z) < \delta. \tag{I}$$

Clearly (I) is true when $n = \kappa$. Now let $n > \kappa$ and suppose that (I) is true for $n - 1$. We will show that (I) is true for n.

Let $z \in X_n \setminus X_{n-1}$, then $z = q_n$. Suppose that $d(x, z) < \delta$ for some $\delta \in\,]0, \delta_0]$.

Then $d(x, z) = d(q_n, x) = d(q_n, b_n) + d(b_n, x) = \varepsilon_n + d(b_n, x)$, so $\varepsilon_n < 1$. Therefore $f(z) = f(q_n) = q_{n+1}$, $f(b_n) = b_{n+1}$ and $\varepsilon_{n+1} \leqslant 2\varepsilon_n$. Hence $d(f(x), f(z)) = d(f(q_n), f(x)) = d(q_{n+1}, b_{n+1}) + d(b_{n+1}, f(x)) = \varepsilon_{n+1} + d(b_{n+1}, f(x))$ (recall that $f(x) \in X_\kappa \subset X_n$).

Since $d(b_n, x) = d(q_n, x) - \varepsilon_n < \delta - \varepsilon_n < \delta$ and $b_n \in X_{n-1}$, by the inductive hypothesis, it follows that $d(f(b_n), f(x)) = d(b_{n+1}, f(x)) < \varepsilon + 2(\delta - \varepsilon_n)$. So $d(f(q_n), f(x)) = \varepsilon_{n+1} + d(b_{n+1}, f(x)) < 2\varepsilon_n + (\varepsilon + 2(\delta - \varepsilon_n)) = \epsilon + 2\delta$.

Now let $z \in Z$ and $n \in \mathbb{N}$ be such that $d(x, z) < \delta_0$, and $z \in X_n$. If $n \leqslant \kappa$, then $z \in X_\kappa$, so $d(f(x), f(z)) < \varepsilon$. Otherwise $d(f(x), f(z)) < \varepsilon + 2\delta_0 < 3\varepsilon$. $\square$

Now we are ready to state our main result.

Theorem 2. *Let f be a continuous self-map on a separable metric space (X, d_0). Then (X, f) admits an isometric chaotic extension (Z, F).*

Proof. Let A be a countable dense subset of (X, d_0), and let us take a set $P = \{p_n\colon n \in \mathbb{N}\}$ consisting of distinct elements such that $X \cap P = \emptyset$.

Let $Z_n = \{[\kappa]_n\colon 0 \leqslant \kappa \leqslant n - 1\}$ be the set of integers modulo n for every $n \in \mathbb{N}$ and set $c_n = [1]_n$. We may assume that $(\bigcup_{n\in\mathbb{N}} Z_n) \cap (X \cup P) = \emptyset$.

Set $Z = X \cup P \cup (\bigcup_{n\in\mathbb{N}} Z_n)$ and let $F : Z \to Z$ be the map given by $F|X = f$, $F(p_n) = p_{n+1}$ and $F([\kappa]_n) = [\kappa + 1]_n$ for every $n \in \mathbb{N}$ and $\kappa \in \{0, \dots, n - 1\}$.

Now let $C = \{c_n\colon n \in \mathbb{N}\}$ and let $t : \mathbb{N} \to A \cup C$ be an ω-to-one map (i.e., every fibre of t is denumerable) such that $m < n$ whenever $c_m = t(n)$. For abbreviation, let t_n stand for $t(n)$.

For every $i \in \mathbb{N}$ and $j \in \{1, \dots, i\}$ set

(i) $q_{i(i-1)+j} = p_{\frac{i(i-1)}{2}+j}$,
(ii) $q_{i^2+j} = F^{j-1}(c_i) = [j]_i$,
(iii) $b_{i(i-1)+j} = F^{j-1}(t_i)$,
(iv) $b_{i^2+j} = p_j = F^{j-1}(p_1)$,
(v) $\varepsilon_{i(i-1)+j} = \varepsilon_{i^2+j} = 2^{j-i}$.

Now set $X_0 = X$, $X_n = X \cup \{q_1, \dots, q_n\}$ for every integer $n \in \mathbb{N}$ and observe that $Z = \bigcup_{n\in\mathbb{N}_0} X_n$.

Let us show that $b_n \in X_{n-1}$ for every $n \in \mathbb{N}$. This is clear if $n = 1$. In the other cases it is enough to show that $m < n$ whenever $b_n = q_m$. This is easy if $b_n = q_m = p_\kappa$, in such a case we have $m < 2\kappa$ and $n > 2\kappa - 1$, so $n > m$. If $b_n = q_m = q_{i^2+j} = F^{j-1}(c_i)$, then $b_n = b_{r(r-1)+\kappa} = F^{k-1}(t_r)$, with $t_r = c_i$ (so $i < r$) and $\kappa \geqslant j$. Therefore $m = i^2 + j < r(r-1) + j \leqslant r(r-1) + \kappa = n$.

We claim that the map F considered as a self-map on Z, endowed with the metric d described before Lemma 1, is continuous.

In fact observe that $f(q_n) = q_{n+1}$, $f(b_n) = b_{n+1}$ and $\varepsilon_{n+1} = 2\varepsilon_n$ whenever $n = i^2 + j$ or $n = i(i-1) + j$, with $1 \leqslant j < i$, while $\varepsilon_n = 1$ in the remaining cases. Moreover $F|X = f$ is continuous, so by Lemma 1, F is continuous.

Now it remains to show that the isometric extension (Z, F) of (X, f) is chaotic.

First let us show that F is topologically transitive. Since Z is a perfect metric space, it is enough to show that the orbit P of p_1 under F is dense in Z.

Let $\varepsilon > 0$. If $x \in A$, let us take i large enough such that $2^{1-i} < \varepsilon$ and $t_i = x$. Then $q_{i(i-1)+1} \in P$, $t_i = b_{i(i-1)+1}$ and $d(q_{i(i-1)+1}, x) = d(q_{i(i-1)+1}, b_{i(i-1)+1}) = 2^{1-i} < \varepsilon$.

If $x = F^{\kappa-1}(c)$ for some $c \in C$ and $\kappa \in \mathbb{N}$, let $i > \kappa$ be such that $2^{\kappa-i} < \varepsilon$ and $t_i = c$. Then $q_{i(i-1)+\kappa} \in P$, $x = F^{\kappa-1}(t_i) = F^{\kappa-1}(c) = b_{i(i-1)+\kappa}$ and $d(q_{i(i-1)+\kappa}, x) = d(q_{i(i-1)+\kappa}, b_{i(i-1)+\kappa}) = 2^{\kappa-i} < \varepsilon$.

Observe that $\bigcup_{n\in\mathbb{N}} Z_n$ consists of periodic points. To check that F is periodically dense, let us show that $P \subset \overline{\bigcup_{n\in\mathbb{N}} Z_n}$.

So let $x = p_n \in P$ and $\varepsilon > 0$. Let us take $i > n$ such that $2^{n-i} < \varepsilon$. Then $q_{i^2+n} \in Z_i$ and $d(p_n, q_{i^2+n}) = d(b_{i^2+n}, q_{i^2+n}) = \varepsilon_{i^2+n} = 2^{n-i} < \varepsilon$.

The proof is complete. □

Sensitive dependence on initial conditions is generally considered as a fundamental feature of any reasonable concept of chaos. In the following remark we will show that every dynamical system has a cofinitely sensitive isometric extension. Cofinite sensitivity is a very strong form of sensitivity: a dynamical system (X, f) is cofinitely sensitive if there is some $\delta > 0$ such that, for any $x \in X$ and any neighbourhood V of x in (X, d), there exist $y \in V$ and some $\kappa \in \mathbb{N}_0$ for which $d(f^n(x), f^n(y)) > \delta$ for every $n > \kappa$ (see [7]).

Remark 3. Let f be a continuous self-map on a metric space (X, d). Let us take $Z = X \times [0, +\infty)$ endowed with the metric ρ given by: $\rho((x,t),(y,s)) = |t - s|$ whenever $x = y$ and $\rho((x,t),(y,s)) = d(x,y) + s + t$ otherwise.

If $p \in Z$ and $\varepsilon > 0$, let us set $B(p, \varepsilon) = \{z \in Z\colon\ \rho(p, z) < \varepsilon\}$.

Let $F : (Z, \rho) \to (Z, \rho)$ be the map given by $F(x, t) = (f(x), 2t)$ for every $(x, t) \in Z$.

Let $p = (x, t) \in Z$, to verify the continuity of F at p it is enough to consider the following facts:

(i) if $t \neq 0$, observe that $F(B(p, \frac{\varepsilon}{2})) \subset B(F(p), \varepsilon)$ for every $\varepsilon \in]0, t[$;
(ii) if $t = 0$ and $\varepsilon > 0$, let $\delta \in]0, \frac{\varepsilon}{4}[$ be such that $d(f(x), f(y)) < \frac{\varepsilon}{2}$ whenever $d(x, y) < \delta$, then $F(B(p, \delta)) \subset B(F(p), \varepsilon)$.

So (Z, F) is an isometric extension of (X, f).

It remains to show that (Z, F) is cofinitely sensitive. Let us take $\delta = 1$, and let $p = (x, t) \in Z$ and $\varepsilon > 0$. Let $\eta \in]0, \varepsilon[$, so $q = (x, t + \eta) \in B(p, \varepsilon)$. Then $\rho(F^n(p), F^n(q)) = 2^n \eta$. Therefore $\rho(F^n(p), F^n(q)) > 1$ for every $n > \log_2(\frac{1}{\eta})$.

Remark 4. Our Theorem 2 relies heavily on the separability of X. It would be interesting to know if there are isometric chaotic extensions even in the non-separable case. Moreover, as pointed out by the referee, one could ask, whenever X is compact, if there is a compact extension space Z.

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